Robust exponential binary pattern storage in Little-Hopfield networks

Christopher Hillar*, Ngoc M. Tran*, and Kilian Koepsell

***Abstract*—The Little-Hopfield network is an auto-associative computational model of neural memory storage and retrieval. This model is known to robustly store collections of randomly generated binary patterns as stable-points of the network dynamics. However, the number of binary memories so storable scales linearly in the number of neurons, and it has been a long-standing open problem whether robust exponential storage of binary patterns was possible in such a network memory model. In this note, we design elementary families of Little-Hopfield networks that solve this problem affirmatively.**

I. Introduction

Inspired by early work of McCulloch-Pitts [1] and Hebb [2], the Little-Hopfield model [3], [4] is a distributed neural network architecture for binary memory storage and denoising. In [4], Hopfield showed experimentally, using the *outer-product learning rule* (OPR), that $.15n$ binary patterns (generated uniformly at random) can be robustly stored in such an n-node network if some fixed percentage of errors in a recovered pattern were tolerated. Later, it was verified that this number was a good approximation to the actual theoretical answer [5]. However, pattern storage without errors in recovery using OPR is provably limited to $n/(4 \log n)$ patterns [6], [7]. Since then, improved methods to fit Little-Hopfield networks more optimally have been developed [8], [9], [10], with the most recent being [11]. Independent of the method, however, arguments of Cover [12] can be used to show that the number of (randomly generated) patterns storable in a Little-Hopfield network with n neurons is at most $2n$, although the exact value is not known (it is $\approx 1.6n$ from experiments in [11]).

Nonetheless, theoretical and experimental evidence suggest that Little-Hopfield networks usually have exponentially many stable-states (i.e., fixed-points of the dynamics). For instance, choosing weights for the model randomly (from a normal distribution) produces an n-node network with $\approx 1.22^n$ fixed-points asymptotically [13], [14], [15]. However, a stored pattern corrupted by only a few bit errors does not typically converge under the network dynamics to the original.

To make precise mathematically the notion of large error tolerance, we say that a sequence B_n of binary pattern collections is *robustly stored* by n-node Little-Hopfield networks if a pattern in B_n having αn of its bits altered at random can be recovered (with probability limiting to 1 as $n \to \infty$) by converging the network dynamics, where $0 < \alpha < 1$ is a constant independent of n. In this sense, the randomly generated networks as discussed above do not have robust storage because the number of bits of corruption tolerated in memory recovery does not increase with the number of nodes.

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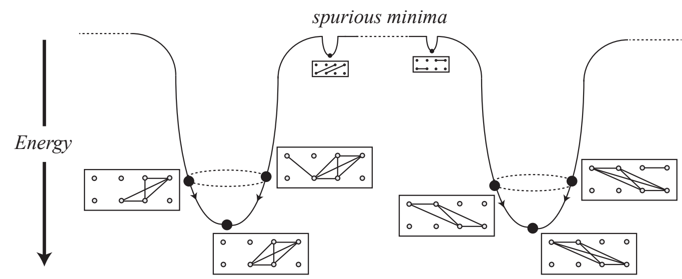


Fig. 1. **Illustration of the energy landscape** of a Little-Hopfield network depicting the robust storage of all 4-cliques in graphs on $v = 8$ vertices. The network dynamics sends a graph that is almost a clique to a graph with smaller energy, until finally converging to the underlying 4-clique as a stable-point.

Another limitation of random networks is that stable-states are difficult to determine from the network parameters. In [16], a Little-Hopfield network with identical weights was shown to have exponential storage on $2n$ nodes, the stored collection consisting of binary vectors with exactly half of their bits equal. Thus, it is possible to design a network with a prescribed exponential number $\binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}}$ of patterns. However, such a network is not able to denoise a single bit of corruption. In particular, this collection of memories is not stored robustly.

Very recently, more sophisticated (non-binary) discrete networks have been developed [17], [18] that give exponential memory storage. However, the storage in these networks is not known to be robust. Moreover, determining or prescribing the network parameters for storing these exponentially many memories is non-trivial (the ideas involve expander codes/graphs and solving linear equations over the integers).

In this note, we design Little-Hopfield networks that robustly store an exponential number of binary patterns. Moreover, our construction is elementary. Two concepts of discrete mathematics are significant players in our development: *cliques in graphs* and *groups of permutations*. We review this technical material in Section II. Full statements of our results appear in Section III, with proofs outlined in Section V. Some preliminary applications are also presented in Section IV.

II. Technical Background

A. Permutation groups

In abstract algebra, a *group* is a set G with a multiplication (or product) $a \circ b$ between elements $a, b \in G$ satisfying the

following three assumptions. We have (i) *associativity of the product*: $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$; (ii) a *multiplicative identity*: there is a unique element $1 \in G$ with $a \circ 1 = 1 \circ a = a$ for all $a \in G$; and (iii) *existence of inverses*: for all $a \in G$, there exists $a^{-1} \in G$ with $a \circ a^{-1} = a^{-1} \circ a = 1$.

Groups are basic but fundamental objects in mathematics. For instance, the set of positive real numbers $\mathbb{R}_{>0}$ forms a group under multiplication. The set of integers $\mathbb{Z}$ also forms a group, but with addition as the group product (and with 0 as the identity element). An important family of non-commutative groups are the $n \times n$ invertible matrices GL_n with entries in the reals $\mathbb{R}$ (the product being ordinary matrix multiplication).

Fix a positive integer v. The set of bijections from the integers $V = \{1, \ldots, v\}$ to themselves are called the *permutations* S_v of V. The set of permutations S_v has size $v! = v \cdot (v-1) \cdots 1$ and forms a group with composition of functions as the product. Sometimes permutations are displayed with two rows that indicate the bijection. For instance, the permutation $\sigma \in S_5$ mapping the numbers $(1, 2, 3, 4, 5)$ bijectively to $(2, 1, 3, 4, 5)$ and its inverse σ^{-1} can be represented:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}, \ \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}. \tag{1}$$

We remark that S_v can be identified naturally as a subgroup of GL_v; it is the set of $v \times v$ permutation matrices in GL_v.

Permutation groups appear frequently in mathematics and its applications. One notable early example is the development of Galois theory which uses the theory of S_5 to deduce the Abel-Ruffini Theorem. This result says that the general fifth degree equation does not have closed-form solutions (e.g., there is no complex number x expressible "in terms of radicals" solving $x^5 + 2x + 1 = 0$). In contrast, equations up to degree four are known to have such explicit solutions.

B. Little-Hopfield networks

Mathematically, a Little-Hopfield network $\mathcal{H} = (\mathbf{J}, \theta)$ on n nodes (e.g. neurons) $\{1, \ldots, n\}$ consists of a real symmetric *weight matrix* $\mathbf{J} = \mathbf{J}^\top \in \mathbb{R}^{n \times n}$ with zero diagonal and a *threshold vector*[1] $\theta \in \mathbb{R}^n$. The possible *states* of the network are all length n binary strings $\{0, 1\}^n$, which we represent as binary column vectors $\mathbf{x} = (x_1, \ldots, x_n)^\top$, each $x_i \in \{0, 1\}$ indicating the state x_i of node i. Given any state $\mathbf{x}$, one (asynchronous) *update of the dynamics* on $\mathbf{x}$ consists of replacing each x_i in $\mathbf{x}$ (in consecutive order starting with $i = 1$) with the value

$$x_i = H(\mathbf{J}_i^\top \mathbf{x} - \theta_i) = H(\sum_{j \neq i} J_{ij} x_j - \theta_i). \tag{2}$$

Here, $\mathbf{J}_i$ is the ith column of $\mathbf{J}$ and H is the *Heaviside function* given by $H(r) = 1$ if $r > 0$ and $H(r) = 0$ if $r \leq 0$. (See Fig. 1 in [11] for a detailed examination of a small network).

The *energy* $E_\mathbf{x}$ of a binary pattern $\mathbf{x}$ in a Little-Hopfield network is a (linear) function of network parameters $\mathbf{J}$ and θ:

$$E_\mathbf{x}(\mathbf{J}, \theta) := -\frac{1}{2}\mathbf{x}^\top \mathbf{J} \mathbf{x} + \theta^\top \mathbf{x} = -\sum_{i<j} x_i x_j J_{ij} + \sum_{i=1}^{n} \theta_i x_i, \tag{3}$$

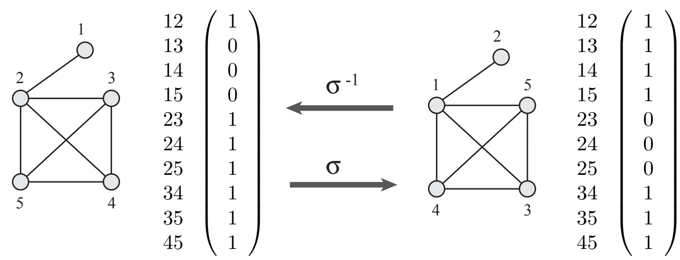
identical to the energy function for an Ising spin glass probabilistic model from statistical physics [19]. In fact, the dynamics of Little-Hopfield networks can be interpreted as 0-temperature Gibbs sampling of this energy function.

A fundamental property of Little-Hopfield networks, observed by Hopfield in [4], is that asynchronous dynamical updates (2) do not increase the energy (3). In particular, one can show that after a finite number of updates, any initial state $\mathbf{x}$ converges to a *fixed-point* (also called stable-point or stored memory) $\mathbf{x}^*$ of the dynamics; that is, $x_i^* = H(\mathbf{J}_i^\top \mathbf{x}^* - \theta_i)$ for each $i = 1, \ldots, n$. Given a binary pattern $\mathbf{x}$, we say more strongly that it is a *strict local minimum* if every $\mathbf{x}'$ with exactly one bit different from $\mathbf{x}$ has a strictly larger energy:

$$0 > E_\mathbf{x} - E_{\mathbf{x}'} = (\mathbf{J}_i^\top \mathbf{x} - \theta_i)\delta_i, \tag{4}$$

where $\delta_i = 1 - 2x_i$ and x_i is the bit that differs between $\mathbf{x}$ and $\mathbf{x}'$. It is straightforward to verify that if $\mathbf{x}$ is a strict local minimum, then it is a fixed-point of the dynamics.

A permutation $\sigma \in S_n$ of the n-nodes of a Little-Hopfield network $\mathcal{H} = (\mathbf{J}, \theta)$ gives rise to another network $\sigma\mathcal{H} = (\sigma\mathbf{J}, \sigma\theta)$, where $\sigma\mathbf{J}$ is the matrix obtained from $\mathbf{J}$ by permuting both its rows and columns by σ, and where $\sigma\theta$ is θ also permuted by σ.



$$\begin{pmatrix} \mathbf{1}_{12} & \mathbf{2}_{13} & \mathbf{3}_{14} & \mathbf{4}_{15} & \mathbf{5}_{23} & \mathbf{6}_{24} & \mathbf{7}_{25} & \mathbf{8}_{34} & \mathbf{9}_{35} & \mathbf{10}_{45} \\ \mathbf{1}_{12} & \mathbf{7}_{25} & \mathbf{5}_{23} & \mathbf{6}_{24} & \mathbf{4}_{15} & \mathbf{2}_{13} & \mathbf{3}_{14} & \mathbf{9}_{35} & \mathbf{10}_{45} & \mathbf{8}_{35} \end{pmatrix}$$

Fig. 2. **Permutations acting on graphs**. A simple graph on $v = 5$ vertices is encoded as a binary vector of length $n = \binom{5}{2} = 10$. Applying the permutation σ in (1) to the vertices $V = \{1, 2, 3, 4, 5\}$ induces a permutation in S_n of the vector encoding the graph, which we also denote σ for notational simplicity.

C. Graphs

A *simple graph* on v vertices $V = \{1, \ldots, v\}$ is represented by a set E of (unordered) pairs of vertices, called the *edges* of the graph. We shall identify graphs on v vertices as binary vectors $\mathbf{x}$ of length $n = \binom{v}{2} = \frac{v(v-1)}{2}$. A coordinate x_e of $\mathbf{x}$ is indexed by an edge $e = \{i, j\}$ $(i < j)$, and is one or zero depending on whether e is contained in the edges of the graph or not (respectively). For simplicity, we list the coordinates in $\mathbf{x}$ lexicographically (i.e., the dictionary order). For $3 \leq k \leq v$, define a *k-clique* to be a graph on v vertices that has edges between each pair of a set of k vertices, but no other edges. There are $\binom{v}{k} = \frac{v(v-1)\cdots(v-k+1)}{k!}$ graphs on v vertices that are k-cliques. The *complete graph* K_v on v vertices is a v-clique.

[1]Throughout this work, vectors such as $\theta = (\theta_1, \ldots, \theta_n)^\top$ will always be represented as columns, where $M^\top$ for a matrix M denotes its transpose.

Relabeling the vertices of a graph is the same as applying a permutation $\sigma \in S_v$ to them. This, in turn, induces a relabeling or permutation of the edges of the graph, which is realized as a permutation of the vector $\mathbf{x}$ representing it; Fig. 2 contains an example. Note that any permutation of the vertices V for a k-clique gives rise to another k-clique.

The storage networks we propose are Little-Hopfield networks with states identified as simple graphs on v vertices. In this case, entries of weight matrices $\mathbf{J} \in \mathbb{R}^{\binom{v}{2} \times \binom{v}{2}}$ are indexed lexicographically by pairs of edges e, f in K_v. A permutation σ on the vertices $V = \{1, \ldots, v\}$ induces a permutation of the edges of a graph, defining a new weight matrix $\sigma\mathbf{J}$, which is the rows and columns of $\mathbf{J}$ permuted accordingly.

III. Main results

Recall the notion of robustness with parameter $\alpha \in (0, 1)$ from the introduction. The following is our first main result.

Theorem 1: For integers $v = 2k$, there is a family of Little-Hopfield networks on $n = \binom{v}{2}$ nodes that robustly store (with parameter $\alpha = 1/2$) all k-cliques in graphs on $2k$ vertices, giving a total number of robustly stored memories on n nodes:

$$\binom{v}{k} \approx \frac{2^{\sqrt{2n}+\frac{1}{4}}}{n^{1/4}\sqrt{\pi}}.$$

Another interpretation of Theorem 1 is that these n-node networks have large numbers of patterns (on the order of $2^{n/2}$ as $n \to \infty$) that converge under the dynamics to a stored binary memory. In other words, the networks have "large basins of attraction" around these stored cliques. For a graphical depiction of one such network, see Fig. 1.

Theorem 1 says that we may store all cliques of a certain fixed size in a Little-Hopfield network. A natural question is whether a range of cliques are so storable as fixed-points of a single network. Our next result answers this question.

Theorem 2: For each integer $v = 2k$, there is a Little-Hopfield network on $n = \binom{v}{2}$ nodes that stores all $2^v(1 - e^{-Cv})$ ℓ-cliques in the range $\frac{2}{D+2}k \le \ell \le \frac{3D+2}{D+2}k$ as strict local minima for constants $C \approx .002$ and $D \approx 13.928$. Moreover, this range stores the most cliques.

We close this section by sketching the main ideas in our proofs. We first show that there is a Little-Hopfield weight matrix storing all k-cliques in some range if and only if there is one which has a simple 3-parameter structure. Note that the set of all J storing a given set of binary patterns as strict local minima is the interior of a (possibly empty) *convex polyhedron* (a finite intersection of closed half-spaces in Euclidean space).

Also, as discussed in Section II, the symmetric group S_v acts on weight matrices J. Consider now the average of J over the group of permutations:

$$J^*(x, y, z) := \frac{1}{v!} \sum_{\sigma \in S_v} \sigma J. \tag{5}$$

The matrix J^* in (5) is *invariant* under the action of S_v; that is, we have

$$\tau J^* = \frac{1}{v!} \sum_{\sigma \in S_v} \tau\sigma J = J^*,$$

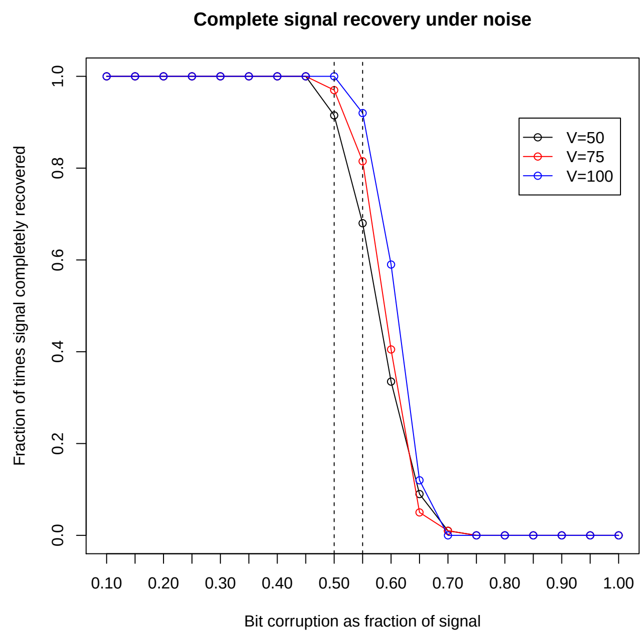


Fig. 3. **One update of clean-up dynamics exhibits robustness with $\alpha = \frac{1}{2}$.** We demonstrate that the exponential number of stored cliques in our networks have large basins of attraction. For each vertex size $v = 2k = 50$, 75, 100, we constructed a Little-Hopfield network storing all k-cliques as fixed-points of the dynamics. Each such k-clique is represented as a binary vector of length $k(2k-1)$. We then corrupted 200 (chosen uniformly at random) k-cliques by changing a fixed percentage of their bits at random and ran the network dynamics on each for one update step (i.e., a pass through all neurons once). The plot shows the percentage of the 200 cliques that were correctly recovered (exactly) as a function of the percent of the pattern that was corrupted. For example, a network with $v = 100$ vertices robustly stores $\binom{100}{50} \approx 10^{29}$ memories (i.e., all 50-cliques in a 100-node graph) using binary vectors of length 4950, each having $\binom{50}{2} = 1225$ nonzero coordinates. In this case, the figure shows that a 50-clique memory represented with 4950 bits may be recovered by the dynamics after flipping 2475 of these bits at random.

since the function from S_v to itself mapping $\sigma \mapsto \tau\sigma$ (for any fixed $\tau \in S_v$) is a bijection.[2]

It is straightforward to check that acting by such a permutation on a Little-Hopfield network that stores all k-cliques as strict local minima will preserve that property. And since the set of all such networks is convex, the convex combination J^* in (5) stores all k-cliques as strict local minima if J does. One now observes that J^* has only 3 free parameters, and the remainder of the argument consists of optimizing these parameters to determine networks that store ranges of cliques.

We remark that "averaging over the group," as is done in (5), occurs frequently in mathematics. For instance, it features prominently in Hilbert's work on invariant theory in algebra, the construction of Haar measures in functional analysis, and in representation theory, more generally. We defer mathematical proof of robustness to future work, but see Fig. 3 for its experimental verification.

[2] An injective function from a finite set to itself is bijective. Thus, we only need to verify injectivity (i.e., $\tau\sigma_1 = \tau\sigma_2$ implies $\sigma_1 = \sigma_2$). But if $\tau\sigma_1 = \tau\sigma_2$, then $\sigma_1 = \tau^{-1}\tau\sigma_1 = \tau^{-1}\tau\sigma_2 = \sigma_2$ so that the map is injective.

IV. Applications

Applications to neuroscience. The Little-Hopfield network is a model of emergent neural computation [3], [4], [20]. One interpretation of the local[3] dynamics in such a model is that by minimizing an energy, the network tries to determine the most probable memory conditioned on a noisy or corrupted version. This concept is in line with arguments of several researchers in theoretical neuroscience [21], [22], [23], [24], [25], [26], [27], [28], and can be traced back to Helmholtz [29]. In addition, recent analyses of spike distributions in neural populations have shown that their joint statistics can sometimes be well-described by the Ising model [30], [31], [32], [33]. The now demonstrated ability of these networks to store large numbers of patterns robustly suggests that the Little-Hopfield architecture should be studied more fully as a possible explanation of neural circuit computation.

Applications to computer science. The networks described in this note have potential implications for several algorithmic problems at the intersection of discrete mathematics, probability, computer science, and machine learning. For instance, a classical NP-complete problem is to determine large cliques in graphs, the so-called MAXCLIQUE problem. We have demonstrated here that when a clique is planted into a empty graph and then “hidden” by turning edges on and off at random, it is still possible to recover the original clique by converging the local dynamics of Little-Hopfield networks. See [34] for the most recent results on this problem.

Applications to coding theory. Our networks also gives rise to new approaches for constructing and working with binary codes. For instance, our networks are easily parallelizable and have similar robustness properties to the well-known optimal codes of Reed-Solomon [35], which use the mathematical machinery of polynomial rings over finite fields.

V. Proofs of theoretical results

Consider the complete graph K_v on v vertices which has $n = \binom{v}{2}$ edges, and fix $k \geq 3$. As discussed in Section II, a binary vector $\mathbf{x} \in \{0,1\}^n$ is identified with a graph $G_{\mathbf{x}}$ on v vertices, where $\mathbf{x}_e = 1$ if edge e is present in the graph. Let $\mathcal{C}_k := \{\mathbf{x} \in \{0,1\}^n : G_{\mathbf{x}} \text{ is a } k\text{-clique}\}$ denote the set of edge vectors representing k-cliques. Identify[4] each Little-Hopfield network with its symmetric weight matrix J. Consider the *3-parameter family* of symmetric matrices $J \in \mathbb{R}^{n\times n}$:

$$J_{ef} = \begin{cases} x & \text{if} \quad |e \cap f| = 1 \\ y & \text{if} \quad |e \cap f| = 0 \\ z & \text{if} \quad e = f, \end{cases}$$

for some $x, y, z \in \mathbb{R}$, where $|e \cap f|$ is the number of vertices that the edges e and f share.

Let $\mathcal{H}_k$ denote the set of Little-Hopfield networks J which store all k-cliques $\mathcal{C}_k$ as strict local minima. We claim that there exists a network J storing all k-cliques if and only if there exists a Little-Hopfield network in the 3-parameter family above storing all k-cliques. Also, let $\mathcal{H}_k^{\Sigma}$ denote the *central cone*, which is the set of all matrices constructed by averaging as in (5) elements of $\mathcal{H}_k$.

[3]The term “local” here refers to the fact that an update (2) to a neuron only requires the feedforward inputs from its neighbors.

[4]For expositional simplicity and without loss of generality, we move the threshold vector θ into the diagonal of the weight matrix since the energy (3) is unchanged by sending parameters $(\mathbf{J}, \theta) \mapsto (\mathbf{J} + 2\text{diag}(\theta), 0)$, where $\text{diag}(\theta)$ is the diagonal matrix with θ along the diagonal.

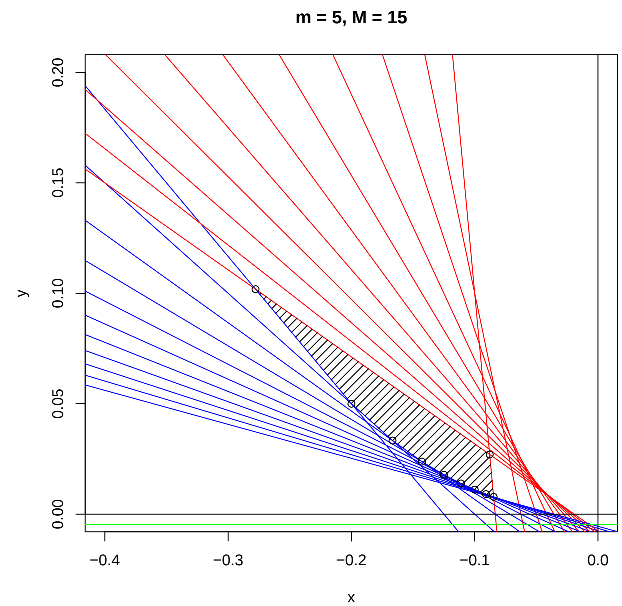


Fig. 4. **Feasible region for network parameters giving exponential storage**. The shaded region is the feasible polygon for network parameters giving clique storage for the range $5 \leq k \leq 15$. Black points are its vertices, and the red, blue, and green lines are the linear constraints.

Proposition 1: The polyhedral cone $\mathcal{H}_k$ is non-empty if and only if its central cone $\mathcal{H}_k^{\Sigma}$ is non-empty. Moreover, $\mathcal{H}_k^{\Sigma}$ is non-empty, and $J(x,y,z) \in \mathcal{H}_k^{\Sigma}$ if and only if its parameters (x, y, z) give the following vector all positive entries:

$$\begin{pmatrix} 4(k-2) & (k-2)(k-3) & -2 \\ -2(k-1) & -(k-1)(k-2) & 2 \\ 0 & -k(k-1) & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad (6)$$

Proof: The cone $\mathcal{H}_k$ is closed under the action of permuting the labels of the vertices of the complete graph; that is, $\mathcal{H}_k$ is an *orbitope* in the sense of Sanyal, Sottile and Sturmfels [36]. The 3-parameter family of matrices above is precisely the set of symmetric matrices invariant under this action. This proves the first claim. The second follows by direct computation. ■

Theorem 3 (Range storage): Fix m, M such that $3 \leq m < M < v$. The set $\bigcap_{k=m}^{M} \mathcal{H}_k$ of Little-Hopfield $\binom{v}{2}$-node networks J storing all k-cliques for $m \leq k \leq M$ is non-empty if and only if (m, M) solve the implicit equation $x_M - x_m < 0$, where

$$x_m = \frac{-(4m - \sqrt{12m^2 - 52m + 57} - 7)}{2(m^2 - m - 2)},$$

$$x_M = \frac{-(4M + \sqrt{12M^2 - 52M + 57} - 7)}{2(M^2 - M - 2)}.$$

In particular, a solution (m, M) is independent of v.

Proof: By Proposition 1, the intersection $\bigcap_{k=m}^{M} \mathcal{H}_k$ is non-empty if and only if the intersection of their central

cones $\bigcap_{k=m}^{M} \mathcal{H}_k^{\Sigma}$ is non-empty. For $J(x, y, z) \in \bigcap_{k=m}^{M} \mathcal{H}_k^{\Sigma}$, its parameters (x, y, z) need to satisfy the system of linear equations (6) for all $m \le k \le M$. Solving this system gives the above constraints on m and M. ■

Note that the intersection of $\bigcap_{k=m}^{M} \mathcal{H}_k^{\Sigma}$ with the plane $z = -0.5$ is a polygon in $\mathbb{R}^2$. We display this polygon in Fig. 4 for $(m, M) = (5, 15)$. Each k adds a triple of red, blue, and green lines, corresponding to the three linear constraints in (6). Note that the green constraints and all but two red constraints are inactive. Vertices of this polygon are the intersections of pairs of blue lines with parameters $k, k+1$, and the two most extreme red lines with parameters $k = m$ and $k = M$.

For large m, we have $x_M - x_m < 0$ when $M \lesssim \frac{2+\sqrt{3}}{2-\sqrt{3}} m \approx 13.9282m$, and it is straightforward to translate this into the statement of Theorem 2 from Section III using basic facts about limiting binomial distributions and their normal approximation.

VI. Acknowledgments

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